

CODIMENSION TWO COMPLETE INTERSECTIONS AND HILBERT-POINCARÉ SERIES

GABRIEL STICLARU

ABSTRACT. We investigate the relation between codimension two smooth complete intersections in a projective space and some naturally associated graded algebras. We give some examples of log-concave polynomials and we propose two conjectures for these algebras.

1. INTRODUCTION

Let $S = \mathbb{C}[x_0, \dots, x_n]$ be the graded ring of polynomials in x_0, \dots, x_n with complex coefficients and denote by S_r the vector space of homogeneous polynomials in S of degree r . For any polynomial $f \in S_r$ we define the *Jacobian ideal* $J_f \subset S$ as the ideal spanned by the partial derivatives f_0, \dots, f_n of f with respect to x_0, \dots, x_n .

The Hilbert-Poincaré series of a graded S -module M of finite type is defined by

$$(1.1) \quad HP(M)(t) = \sum_{k \geq 0} \dim M_k \cdot t^k$$

and it is known, to be a rational function of the form

$$(1.2) \quad HP(M)(t) = \frac{P(M)(t)}{(1-t)^{n+1}} = \frac{Q(M)(t)}{(1-t)^d}.$$

For any polynomial $f \in S_r$ we define the corresponding graded *Milnor* (or *Jacobian*) algebra by

$$(1.3) \quad M(f) = S/J_f.$$

The hypersurface $V(f) : f = 0$ is smooth of degree d if and only if $\dim M(f) < \infty$, and then it is known that

$$(1.4) \quad HP(M(f))(t) = \frac{(1-t^{d-1})^{n+1}}{(1-t)^{n+1}}.$$

Our examples and the second conjecture refer to log-concavity and unimodal property. Before listing these examples we recall some basic notions.

A sequence a_0, a_1, \dots, a_m of real numbers is said to be *log-concave* (resp. *strictly log-concave*) if it verifies $a_k^2 \geq a_{k-1}a_{k+1}$ (resp. $a_k^2 > a_{k-1}a_{k+1}$) for $k = 1, 2, \dots, m-1$. An infinite sequence a_k , $k \in \mathbb{N}$ is (strictly) log-concave if any truncation of it is (strictly) log-concave.

2010 *Mathematics Subject Classification.* 13D40, 14J70, 14Q10, 32S25.

Key words and phrases. projective hypersurfaces, singularities, Milnor algebra, Hilbert-Poincaré series, log concavity, free resolutions.

Such sequences play an important role in Combinatorics and Algebraic Geometry, see for instance the recent paper [4]. Recall that a sequence a_0, \dots, a_m of real numbers is said to be *unimodal* if there is an integer i between 0 and m such that

$$a_0 \leq a_1 \leq \dots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \dots \geq a_m.$$

A nonnegative log-concave sequence with no internal zeros (i.e. a sequence for which the indices of the nonzero elements form a set of consecutive integers), is known to be unimodal, see [4] and the references there.

A polynomial $P(t) \in \mathbb{R}[t]$, $P(t) = \sum_{j=0}^{j=m} a_j t^j$, with coefficients a_j for $0 \leq j \leq m$, is said to be log-concave (resp. unimodal) if the sequence of its coefficients is log-concave (resp. unimodal). For other examples with log-concave polynomials, see [5]. For Singular language and applications, see [2] and [3].

2. MAIN RESULTS

Assume from now on that $V(f)$ is smooth of degree d and let $g \in S_e$ be another homogeneous polynomial of degree e . We say that $V(f, g) = V(f) \cap V(g)$ is a smooth complete intersection if any point $p \in V(f, g)$ is smooth on both $V(f)$ and $V(g)$ and the corresponding tangent spaces $T_p V(f)$ and $T_p V(g)$ are distinct. Our first result is the following.

Theorem 2.1. *Let $m_2(f, g)$ be the ideal generated by all the 2×2 minors in the Jacobian matrix of the mapping $(f, g) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^2$. Define $A(f, g) = S/((f) + m_2(f, g))$ and $B(f, g) = S/((f) + (g) + m_2(f, g))$.*

With this notation, the following conditions are equivalent.

- (i) $V(f, g) = V(f) \cap V(g)$ is a smooth complete intersection.
- (ii) $\dim A(f, g) < \infty$.
- (iii) $\dim B(f, g) < \infty$.

If $f = x_0$, then $A(f, g) = B(f, g) = S'/J_{g'} = M(g')$, where $S' = \mathbb{C}[x_1, \dots, x_n]$ and $g'(x_1, \dots, x_n) = g(0, x_1, \dots, x_n)$. However, in general we have the following.

Proposition 2.2. *With the above notation, the ideals $I(f, g) = (f) + m_2(f, g)$ and $J(f, g) = (f) + (g) + m_2(f, g)$ have the same radical, but are distinct in general. When the equivalent conditions of Theorem 2.1 hold, this common radical is the maximal ideal (x_0, \dots, x_n) .*

Proof. (Theorem 2.1 and Proposition 2.2).

The equivalence of (i) and (iii) in Theorem 2.1 is a classical fact, see for instance Proposition (6.39) in [1]. To prove the equivalence of (ii) and (iii) in Theorem 2.1, it is enough to prove the first part of the claim in Proposition 2.2, namely that the ideals $I(f, g)$ and $J(f, g)$ have the same radical. To do this, it is enough by Hilbert's Nulstellensatz, see for instance Theorem (2.14) in [1] to show that the corresponding zero sets $V(I(f, g))$ and $V(J(f, g))$ in \mathbb{P}^n coincide. Moreover, it is clear that $V(J(f, g)) \subset V(I(f, g))$.

Conversely, note that a point p in $V(I(f, g))$ satisfies $f(p) = 0$ and the differentials $df(p)$ and $dg(p)$ are proportional. Since $df(p) \neq 0$ as $V(f)$ is supposed to be smooth, it follows that there is $\lambda \in \mathbb{C}$ such that $dg(p) = \lambda df(p)$. This implies via the Euler formula that $g(p) = 0$, hence $p \in V(J(f, g))$.

The second part of the claim in Proposition 2.2 is dealt with in the next section by means of examples computed using Singular. □

Remark 2.3. Note that the equivalent condition in Theorem 2.1 do not imply $\dim S/m_2(f, g) < \infty$. Take for instance $f = x_0$ and $g = x_0^2 + x_1^2 + \dots + x_n^2$.

As a consequence of Conjecture 3.3 we can prove the following formulas for the Hilbert-Poincaré series of the graded S -modules $A(f, g)$ and $B(f, g)$. The result is given in the form $HP(M)(t) = \frac{P(M)(t)}{(1-t)^4}$ and hence it is enough to give the polynomial $P(M)$.

Proposition 2.4. *The Hilbert-Poincaré series of a graded S -module $A(f, g)$ and $B(f, g)$ depend only on the degrees d and e , when the equivalent conditions of Theorem 2.1 hold. More precisely, one has the following in \mathbb{P}^3 .*

$$(i) \ P(A(f, g))(t) = 1 - [t^d + 6t^{d+e-2}] + [4t^{d+2e-3} + 4t^{2d+e-3} + 6t^{2d+e-2}] - [t^{d+3e-4} + t^{2d+2e-4} + 4t^{2d+2e-3} + t^{3d+e-4} + 4t^{3d+e-3}] + [t^{2d+3e-4} + t^{3d+2e-4} + t^{4d+e-4}].$$

$$(ii) \ P(B(f, g))(t) = 1 - [t^e + t^d + 6t^{d+e-2}] + [4t^{d+e-1} + 4t^{d+2e-3} + 4t^{2d+e-3}] - [t^{d+3e-4} + t^{2d+2e-4} + 4t^{2d+2e-3} + t^{3d+e-4}] + [t^{2d+3e-4} + t^{3d+2e-4}].$$

Proof. We explain now briefly how to get Proposition 2.4 from Conjecture 3.3.

To get the formulas, we start with the resolution and get

$$HP(M)(t) = HP(S)(t) - HP(R_1)(t) + HP(R_2)(t) - HP(R_3)(t) + HP(R_4)(t),$$

where $HP(S)(t) = \frac{1}{(1-t)^4}$.

Then we use the well-known formulas $HP(N_1 \oplus N_2)(t) = HP(N_1)(t) + HP(N_2)(t)$,

$$HP(N^p(-q))(t) = pt^q HP(N)(t), \quad HP(S(-k))(t) = \frac{t^k}{(1-t)^4}.$$

□

Remark 2.5. Note that if $f = x$ and g of degree e , we obtain the Hilbert-Poincaré series for any smooth hypersurfaces of degree e , in \mathbb{P}^2 :

$$S(t) = \frac{(1-t^{e-1})^3}{(1-t)^3} = (1 + t + t^2 + \dots + t^{e-2})^3.$$

3. EXAMPLES AND CONJECTURES

The examples and the conjectures concern codimension 2 complete intersections in \mathbb{P}^3 with coordinates $(x : y : z : w)$.

Example 3.1. The polynomial g does not belong to the ideal $I(f, g)$ in general, i.e. $I(f, g) \neq J(f, g)$. Let $V(f) : f = x^2 + y^2 + z^2 + w^2 = 0$ Fermat smooth surface of degree two and nodal surfaces $2A_1$ type, $V(g) : g = xzw + (z + w)y^2 + x^3 + x^2y + xy^2 + y^3 = 0$.

Indeed $\text{Radical}(I(f,g)) = \text{Radical}(J(f,g)) = \text{Ideal}(x,y,z,w)$. The ideal $I(f,g)$ has seven generators,

$I(f,g) = (G_1, G_2, G_3, G_4, G_5, G_6, G_7)$, where:

$$G_1 = 2x^3 - 2x^2y + 2xy^2 - 2y^3 + 4xyz + 4xyw - 2yzw,$$

$$G_2 = 2xy^2 - 6x^2z - 4xyz - 2y^2z + 2x^2w - 2z^2w,$$

$$G_3 = 2xy^2 + 2x^2z - 6x^2w - 4xyw - 2y^2w - 2zw^2,$$

$$G_4 = 2y^3 - 2x^2z - 4xyz - 6y^2z - 4yz^2 + 2xyw - 4yzw,$$

$$G_5 = 2y^3 + 2xyz - 2x^2w - 4xyw - 6y^2w - 4yzw - 4yw^2,$$

$$G_6 = 2y^2z + 2xz^2 - 2y^2w - 2xw^2,$$

$$G_7 = x^2 + y^2 + z^2 + w^2.$$

One can check the identity: $g = (7/2)G_1 + (1/2)G_2 + (-1/2)G_3 + 2G_4 + (-2)G_5 + 3G_6 + (-6x + 8y + 8z - 8w)G_7 + (-8z^3 - 28xyw + 2y^2w + xzw + 7yzw + 9z^2w + 12xw^2 - 16yw^2 - 9zw^2 + 8w^3)$ where Normal Form (remainder modulo $I(f,g)$) of g is not zero, so g does not belong to $I(f,g)$.

Example 3.2. The Hilbert-Poincaré series of a graded S -module $A(f,g)$ and $B(f,g)$ depend only on the degrees d and e , when the equivalent conditions of Theorem 2.1 hold.

The table below contain singular cubic projective surface in \mathbb{P}^3 displayed with the type of singularity.

Type	polynomial g of degree three
A_2	$(x + y + z)(x + 2y + 3z)w + xyz$
$2A_1$	$xzw + (z + w)y^2 + x^3 + x^2y + xy^2 + y^3$
$A_1 + A_2$	$x^3 + y^3 + x^2y + xy^2 + y^2z + xzw$
A_4	$y^2z + yx^2 - z^3 + xzw$
$3A_1$	$y^3 + y^2(x + z + w) + 4xzw$
$A_1 + A_3$	$wxz + (x + z)(y^2 - x^2)$
A_5	$wxz + y^2z + x^3 - z^3$
D_4	$w(x + y + z)^2 + xyz$
$2A_1 + A_2$	$wxz + y^2(x + y + z)$
$A_1 + A_4$	$wxz + y^2z + yx^2$
D_5	$wx^2 + xz^2 + y^2z$
$4A_1$	$w(xy + xz + yz) + xyz$
$A_1 + 2A_2$	$wxz + xy^2 + y^3$
$2A_1 + A_3$	$wxz + (x + z)y^2$
$A_1 + A_5$	$wxz + y^2z + x^3$
E_6	$wx^2 + xz^2 + y^3$
$3A_2$	$wxz + y^3$
\tilde{E}_8	$y^3 + 2z^3 + 4w^3$

- (1) Let $d = 2, e = 3$, $V(f) : f = x^2 + y^2 + z^2 + w^2 = 0$ and $V(g) : g = 0$ any singular cubic projective surface in \mathbb{P}^3 from the table above. In all cases, g not belongs to $I(f,g)$, $I(f,g)$ and $J(f,g)$ have the same radical $\text{Ideal}(x,y,z,w)$, but $A(f,g)$ and $B(f,g)$ are distinct, with different Hilbert-Poincaré finite series, log-concave polynomials.

$$HP(A(f, g))(t) = 1 + 4t + 9t^2 + 10t^3 + 5t^4 + t^5.$$

$$HP(B(f, g))(t) = 1 + 4t + 9t^2 + 9t^3 + 5t^4 + t^5.$$

- (2) Let $d = 2, e = 3$, $V(f) : f = x^2 + y^2 + z^2 + w^2 = 0$ and $V(g) : g = 0$ any singular cubic projective surface in \mathbb{P}^3 from the list below.

$$A_3 : g = xzw + (x + z)(y^2 - x^2 - z^2) = 0,$$

$$2A_2 : g = x^3 + y^3 + x^2y + xy^2 + xzw = 0.$$

In these cases, g not belongs to $I(f, g)$. $I(f, g)$ and $J(f, g)$ have the same radical other than $\text{Ideal}(x, y, z, w)$, but $A(f, g)$ and $B(f, g)$ are distinct, with different Hilbert-Poincaré infinite series.

$$HP(A(f, g))(t) = 1 + 4t + 9t^2 + 10t^3 + 5t^4 + 2(t^5 + t^6 + \dots),$$

$$HP(B(f, g))(t) = 1 + 4t + 9t^2 + 9t^3 + 5t^4 + 2(t^5 + t^6 + \dots).$$

- (3) Let $d = 3, e = 3$, $V(f) : f = x^3 + y^3 + z^3 + w^3 = 0$ and $V(g) : g = 0$ any singular cubic projective surface in \mathbb{P}^3 from the table above. In all cases, g not belongs to $I(f, g)$, $I(f, g)$ and $J(f, g)$ have the same radical $\text{Ideal}(x, y, z, w)$, but $A(f, g)$ and $B(f, g)$ are distinct, with different Hilbert-Poincaré finite series, log-concave polynomials.

$$HP(A(f, g))(t) = 1 + 4t + 10t^2 + 19t^3 + 25t^4 + 22t^5 + 12t^6 + 3t^7.$$

$$HP(B(f, g))(t) = 1 + 4t + 10t^2 + 18t^3 + 21t^4 + 16t^5 + 8t^6 + 2t^7.$$

- (4) Let $d = 1, e = 3$, $V(f) : f = x = 0$ and $V(g) : g = 0$ any singular cubic projective surface in \mathbb{P}^3 from the table above. We obtain the Hilbert-Poincaré series for any smooth hypersurfaces of degree 3 in \mathbb{P}^2 : $S(t) = \frac{(1-t^2)^3}{(1-t)^3} = 1 + 3t + 3t^2 + t^3$.

These examples motivate our following conjectures.

The best way to understand the graded S -modules $A(f, g)$ and $B(f, g)$ is to construct their free resolutions. Based on the free resolution, we can compute the Hilbert-Poincaré series for each graded algebra. We propose the following conjecture, concern codimension 2 complete intersections in \mathbb{P}^3

Conjecture 3.3. *Let $A(f, g)$ and $B(f, g)$ with $\dim A(f, g)$ and $\dim B(f, g) < \infty$. Then the minimal graded free resolutions of these algebras are the following.*

$$(3.1) \quad 0 \rightarrow R_4 \rightarrow R_3 \rightarrow R_2 \rightarrow R_1 \rightarrow S \rightarrow M \rightarrow 0$$

(i) If $M = A(f, g)$,

$$R_1 = S(-d) \oplus S^6[-(d + e - 2)],$$

$$R_2 = S^4[-(d + 2e - 3)] \oplus S^4[-(2d + e - 3)] \oplus S^6[-(2d + e - 2)],$$

$$R_3 = S[-(d + 3e - 4)] \oplus S[-(2d + 2e - 4)] \oplus S^4[-(2d + 2e - 3)] \oplus S[-(3d + e - 4)] \oplus S^4[-(3d + e - 3)],$$

$$R_4 = S[-(2d + 3e - 4)] \oplus S[-(3d + 2e - 4)] \oplus S[-(4d + e - 4)].$$

(ii) If $M = B(f, g)$,

$$R_1 = S(-e) \oplus S(-d) \oplus S^6[-(d + e - 2)],$$

$$R_2 = S^4[-(d + e - 1)] \oplus S^4[-(d + 2e - 3)] \oplus S^4[-(2d + e - 3)],$$

$$R_3 = S[-(d + 3e - 4)] \oplus S[-(2d + 2e - 4)] \oplus S^4[-(2d + 2e - 3)] \oplus S[-(3d + e - 4)],$$

$$R_4 = S[-(2d + 3e - 4)] \oplus S[-(3d + 2e - 4)].$$

Conjecture 3.4. *When the equivalent conditions of Theorem 2.1 hold, the Hilbert-Poincaré finite series $HP(A(f, g))$ and $HP(B(f, g))$ are log-concave polynomials with no internal zeros, in particular they are unimodal.*

4. CONCLUSION

The main results of this paper are Theorem 2.1 and Proposition 2.2 in \mathbb{P}^n and Proposition 2.4 in \mathbb{P}^3 .

By analogy with Milnor algebra associated with one homogeneous polynomial in \mathbb{P}^n , we define two graded algebras $A(f, g)$ and $B(f, g)$ associated with homogeneous polynomials f and g , where projective hypersurface $V(f) : f = 0$ is smooth. We replace the *Jacobian ideal* with two ideals $I(f, g)$ and $J(f, g)$, generated by f, g and 2×2 minors in the Jacobian matrix (f, g) . For these graded algebras we consider Hilbert-Poincaré series that encapsulates information about the dimensions of homogeneous components. When $f = x_0$, the Hilbert-Poincaré series corresponds to the smooth hypersurface in \mathbb{P}^{n-1} , as in the Milnor algebra case.

The main conclusion of the given examples 3.1 and 3.2 are Conjectures 3.3 and 3.4 in \mathbb{P}^3 .

When the equivalent conditions of Theorem 2.1 hold, the Hilbert-Poincaré series of a graded module $A(f, g)$ and $B(f, g)$ depend only on the degrees of f and g , as in Milnor algebra smooth case, and are log-concave polynomials.

Finally, we consider this construction as a new method to generate many, infinite families of log-concave polynomials.

Acknowledgment

The author is grateful to A. Dimca for suggesting this problem.

REFERENCES

- [1] A. Dimca, *Topics on Real and Complex Singularities*, Vieweg Advanced Lecture in Mathematics, Friedr. Vieweg und Sohn, Braunschweig, 1987. 2
- [2] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, *Singular - A computer algebra system for polynomial computations*. Available at <http://www.singular.uni-kl.de>. 1
- [3] G.-M. Greuel, G. Pfister, *A Singular Introduction to Commutative Algebra* (with contributions by O. Bachmann, C. Lossen, and H. Schnemann). Springer-Verlag 2002, second edition 2007. 1
- [4] J. Huh, *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*. J. Amer. Math. Soc. 25 (2012), 907-927. 1
- [5] G. Sticlaru, *Log-concavity of Milnor algebras for projective hypersurfaces*, arXiv:1310.0506v2. 1

E-mail address: gabrielsticlaru@yahoo.com